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PASSAGE THROUGH THE SPEED OF SOUND IN LAVAL NOZZLES WITH CIRCULAR CROSS SECTION

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The equations

$$-u\,\partial u/\partial x + \partial v/\partial r + v/r = 0, \quad \partial u/\partial r = \partial v/\partial x \tag{1}$$

describing the axisymmetric transonic flow, were obtained by T. von Karman [1]. We shall apply them for the study of the peculiarities of the passage through the speed of sound in Laval nozzles with circular cross sections. With this in view, we shall assign the distribution of the velocity components of particles along the nozzle axis, i. e. at $r = 0$, in the form

$$u = -A_1|x|^k \text{ at } x<0; \quad u = A_2x^k \text{ at } x>0, \quad v = 0$$
$$(A_1>0,\ A_2>0). \tag{2}$$

We shall consider that the values of the indicator k are included in the interval $1 < k < 2$; in the respective gas flows, the acoustic line is the exponential curve, concave or bent on the side of the incident flow. The problem (2) with $k = 1$, was studied in detail in the works [2, 3]. At $k = 2$, the passage , or transitional line becomes straight and perpendicular to the symmetry axis of the nozzle [4 — 6].

If densification discontinuities develop in the flows, the solutions of the equations (1) must satisfy additional boundary conditions at the wave front, apart from the initial data (2), namely

* O PEREKHODE CHEREZ SKOROST' ZVUKA V SOPLAKH LAVALYA S KRUGLYM POPERECHNYM SECHENIYEM.

the shock polar equation [7]

$$2(v_2 - v_3)^2 = (u_2 - u_3)^2 (u_2 + u_3) \qquad (3)$$

and the correlation [5]

$$u_2\, dx_2 / dr + v_2 = u_3\, dx_2 / dr + v_3, \qquad (4)$$

equivalent to the continuity condition of the tangential component of the velocity vector. In the equations (3) and (4), the indices refer to parameters at different sides of the shock front, and $x_2 = x_2(r)$ is the equation giving its position.

It is easy to show that solution of the problem (2), searched for, is self-modeling

$$u = r^{2(n-1)} f(\xi), \quad v = r^{3(n-1)} g(\xi), \quad \xi = x / r^n, \quad n = 2/(2-k),$$

with the shock front equation having the form $\xi = \xi_2 = \text{const.}$

The substitution of written formulas into the system of equations (1) and the exclusion of the function $g(\xi)$ from the correlations obtained, give for the determination of $f(\xi)$ a differential equation of the second order

$$(f - n^2\xi^2)\, d^2f / d\xi^2 + (df / d\xi)^2 + n(3n-4)\,\xi\, df / d\xi - 4(n-1)^2 f = 0. \qquad (5)$$

To simplify the qualitative investigation of the problem considered, we postulate [2, 3]

$$f = \xi^2 F(\eta), \quad dF / d\eta = \Psi, \quad \eta = \ln|\xi|. \qquad (6)$$

In the new variables, the problem's (5) order lowers:

$$d\Psi / dF = (-4F - 4n\Psi - 6F^2 + 7F\Psi + \Psi^2)/(n^2 - F)\,\Psi. \qquad (7)$$

Data (2) of Cauchy lead to the requirement that the integral curve of the equation (7), picturing the field of velocities in the vicinity of nozzle's neck, begin and end in its peculiar point A (0, 0), which corresponds to the axis x. In the vicinity of A this curve is given by the expansion

$$\Psi = -\frac{2}{r} F - 3\,\frac{(n - 4/3)(n-1)}{n^3} F^2 + 3/4\,\frac{(n - 4/3)(n-1)(6n^2 - 7n - 2)}{n^5} F^2 + \ldots \qquad (8)$$

The boundary conditions (3) and (4) in the plane $F\Psi$ will be written

$$F_2 + F_3 = 2n^2, \quad \Psi_2 + \Psi_3 = -2n(7n-4). \qquad (9)$$

The motion along the integral curve in the plane $F\Psi$ from the point A in the direction of the node $C\,[n^2, n(4-7n+\sqrt{25n^2-56n+32})/2]$ describes the gas flow in the inlet part of the nozzle between the axis x and the C_-^0- characteristic arriving at its center. The passing through the point C implies the intersection of the C_-^0-characteristic in the physical plane. By subsequent integration of formulas (6), it is easy to show that the discontinuity in the i-th derivative of the function $\Psi(F)$ corresponds to discontinuities in (i + 1)derivatives of the velocity vector components over the coordinates. That is why the character of flow peculiarity on the C_-^0- characteristic is determined by the expansion of the function $\Psi(F)$ in the vicinity of the point C

$$\Psi = {}^1/_2\, n\,(4-7n+\sqrt{25n^2-56n+32}\,)+$$
$$+\, a_1(F-n^2) + a_2(F-n^2)^2 + \ldots + b_1(F-n^2)^{\lambda} + \ldots \qquad (10)$$

Here the coefficients a_i depend only on n, the constant b_1 is arbitrary, and the exponent λ at the first part of the irregular part is given by the formula

$$\lambda = -2\sqrt{25n^2-56n+32}\,/\,(4-7n+\sqrt{25n^2-56n+32}). \qquad (11)$$

As is well known, the exponent λ is equal to the ratio of the roots of the characteristic equation determining the type of the particular point C. So long as its value is distinct from a whole positive number, only one of the integrals is holomorphic, (10), according to the theorem by [Brio and Buke]* [8]. It is obtained by equating the arbitrary constant b_1 to zero. To the contrary, at integral values of either all the integrals of the ordinary differential equation are representable with the aid of Taylor series, or none of them will be holomorphic. In the latter case, the representation (10) loses its strength, and logarithimic terms appear in it. Analysis of the equation (7) shows, that at whole positive numbers λ there is no holomorphic

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integral at the point C, i. e. the second possibility, acceptable by the strength of the [Brio and Buke]* theorem, is materialized.

Let us plot the flows without peculiarities in the derivatives of the velocity components over the coordinates on the C_-^0-characteristic with the aid of integrals corresponding to mixed values of the exponent . In them, the current lines at intersection points with the C_-^0- characteristic are also devoid of peculiarities, and therefore no artificial approach, imparting a special shape to nozzle walls, is required for the materialization of analytical flows in the vicinity of the C_-^0-characteristic. Calculations, rapidly conducted with the aid of electron computer BESM-2, show, that the construction of the flow field without peculiarities on a characteristic, closing the channel intake, is indeed possible at $k = 1.167$, $n = 2.366$ and $A_2 = 0.629\ A_I$ In such a flow, discontinuities of the third derivatives of velocity vector component spread along the C_+^0 -characteristic emerging from the center. Axisymmetric flows with shock waves emerging at the center of nozzles with smooth walls do not exist.

The flow, realized at $k = 1.157$ and $A_2 = 0.629\ A_1$, just as the flow, analyzed in [2, 3] with $k = 1$ and $A_2 = A_1$, has in the vicinity of the center of the nozzle, an asymptotic character. The peculiarities present in it onset in the flow itself, at the point of intersection of the acoustic line with the symmetry axis, and not in the walls; then they are carried toward the exhaust part of the channel.

The fields of velocities are obtained essentially different in the two indicated types of asymptotic flows. The velocity increases monotonically along each of the current lines in the flow. To the contrary, in a gas flow with $k = 1.157$, the distribution of velocity along the lines of current has two relative extrema: a maximum between the acoustic line and the C_-^0-characteristic, and a minimum in the region included between the $C_\mp^0$ -characteristics. In connection with that, the shape of the curves u = const at $k = 1.157$, will have the form shown in Fig. 1.

In all the remaining continuous as well as discontinuous flows along the C_-^0- characteristic, closing the nozzle's intake, either peculiarities in the derivatives of velocity components propagate along

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the coordinates. In formula (11) we have $\lambda = 1, 2, 3, 4$ respectively at $k=12/17, (7+2\sqrt{21})/14, (26+25\sqrt{2})/41, 3(7+\sqrt{91})/28$ and $n = 17/11, 4(21+2\sqrt{21})/51, (56+25\sqrt{2})/23, 4(35+3\sqrt{91})/29$. When the values $k \to 2$ and $n \to \infty$, the exponent $\lambda \to 5$. Hence, we conclude that at $1 \leqslant k \leqslant (7+2\sqrt{21})/14$ all the derivatives of velocity components, beginning with the third ones, have infinite discontinuities; when $(7+2\sqrt{21})/14 < k \leqslant (26+25\sqrt{2})/41$, the infinite discontinuities arise in the fourth and higher derivatives; at $(26+25\sqrt{2})/41 < k \leqslant 3(7+\sqrt{91})/28$ the infinite discontinuities have the fifths and higher derivatives; finally, when $3(7+\sqrt{91})/28 < k < 2$, all the derivatives of the velocity vector components, beginning with the sixths, undergo infinite discontinuities. At $k = (7+2\sqrt{21})/14, (26+25\sqrt{2})/41, 3(7+\sqrt{91})/28$ the expansion (10) loses its strength and it must be replaced by an expansion containing logarithmic terms.

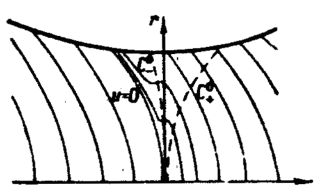

Fig. 1

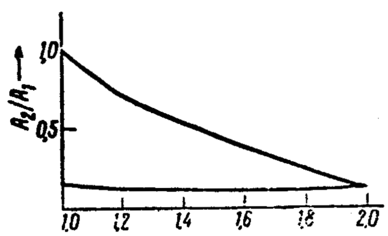

Fig. 2

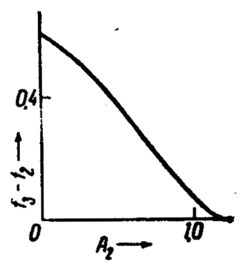

Fig. 3

In order to obtain the fastest expandable flow beyond the C_+^0 - characteristic, it is necessary to effect a jump from the point C in the phase plane F I into the saddle $D[n^2, n(4-7n-\sqrt{25n^2-56n+32})/2]$, and then move along the separatrix passing through it in the direction of the infinitely remote point E, which is situated on the straight line $\Psi = -2F$, and then return again along the extension of the separatrix to the point C. The motion along the indicated integral curve of the equation (7) in the retrograde direction from the point C to point D through E represents the flow from all the continuous ones as the most slowly expanding beyond the C_+^0 – characteristic. The construction of limit flows allows the finding of the region of ratio A_2/A_1 variation corresponding to shockless gas flows. The boundaries of this region are shown in Fig. 2. In flows with maximum expansion in the direction of nozzle's exhaust, disruptions of accelerations are forming at the C_-^0- characteristic, and in those with minimum expansion — at the C_+^0- characteristic.

In discontinuous flows the densification jump arises at the center of the channel and then is carried downward along the flow. For the construction of discontinuous flows we must choose such integral curves of (7), which emerge from the point C in the direction of the point E, and whose extensions, beginning at E, are situated below the separatrix passing through the saddle D. Alomg such curves, the values $I \rightarrow -\infty$ at $F \rightarrow n^2$, and in the corresponding gas flows there emerge boundary lines, being envelopes of the C_+^0- characteristics and bearing infinite values of accelerations. Since the flow with infinite accelerations is physically senseless, a shoc: wave must form in it prior to the emergence of the boundary line. But it appears to be impossible to introduce a shock wave into a flow, where there are no infinite accelerations. Let us note that the gas flow in the nozzle's intake is not disrupted at the onset of the densification jump.

The flow of gas behind the densification jump in variables F, Ψ must be plotted by a portion of the integral curve (8). Together with the equalities (9), this condition defines the intensity of the shock wave. For the computations it is simplest to utilize the straight integration of the initial equation (5), subsequently converting the results in the plane $F\Psi$. As an example, we have shown in Fig. 3 the dependence $f_3 - f_2$ on the constant A_2 at $k = 1.157$. The dependence of the coordinate ξ_2 of the densification jump on the same constant is plotted in Fig. 4.

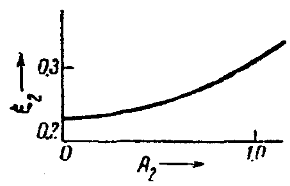

Fig. 4

The quantity A_1 was so selected that the position of the C_-^0-characteristic be determinable by the equality $\xi_1 = -1$. In order to induce the appearance of the shock wave directly at the point of intersection of the acoustic curve with the axis of the circular nozzle, it it necessary to send there perturbations along the C_-^0- characteristic. Truly, however, the indicated perturbations can be very weak, Thus the gas flows with

$k = (7 + 2\sqrt{21})/14,\ (26 + 25\sqrt{2})/41,\ 3(7 + \sqrt{91})/28$ provide examples of shock wave formation as a result of reflection from the center of the nozzle of logarithmic peculiarities respectively in the thirds, fourths and fifths derivatives of velocity components over the coordinates. It must be stressed that the peculiarities, introduced to the center along the C_- - characteristic, do not have any essential significance for the analysis of the causes of shock wave formation in the vicinity of the restriction (narrowing) of the channel. Shock wave arise as a result of appearance of an enveloppe at characteristics emerging from points of the acoustic line and inclined downward along the flow. If the field of velocities preserves its analytical character at crossing the C_- -characterustic, but in the whole corresponds to the nozzle, in which a continuous flow cannot be achieved, the densification jump onsets somewhat to the right of the transitional line in the supersonic region.

It follows from Fig. 2, that shock waves form in the flows only when the values of the ratio A_2/A_1 becomes lower than a specific limit. Hence, it is easy to obtain, that constructively, the formation of shock waves near the critical cross section of the nozzle is linked with too prolonged transitional part. Thus, when constructing nozzles, the transitional part should be made as short as possible; the increase of distance between the neck and the intake of the channel leads to a slower expansion of the flow, and, in the end, to the appearnace of discontinuities. In the limitcase, the velocity behind the densification jump is equal in magnitude to the critical one and is directed along the axis of symmetry of the nozzle.

**** THE END ****

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